Entanglement conditions for two-mode states

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We provide a class of inequalities whose violation shows the presence of entanglement in two-mode systems. We initially consider observables that are quadratic in the mode creation and annihilation operators and find conditions under which a two-mode state is entangled. Further examination allows us to formulate additional conditions for detecting entanglement. We conclude by showing how the methods used here can be extended to find entanglement in systems of more than two modes.

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Entanglement has proven to be a valuable resource in quantum information processing. However, determining whether or not a state is entangled is often far from simple. Methods such as the Peres-Horodecki positive partial transpose condition [1], entanglement witnesses [2], and hierarchies of entanglement conditions [3] exist, but are not always straightforward to apply. In particular, for systems with continuous degrees of freedom, such as particle position or momentum or the quadrature components of field modes, the number of available criteria for detecting entanglement is very limited. Each of the known criteria detects only a subset of the set of entangled states. In many cases, these criteria are in the form of inequalities [4]-[9]. In general they provide only sufficient conditions for detecting entanglement [10]. The utility of most of these inequalities is however limited for non-Gaussian bipartite states. For example, none of these conditions can detect the fact that the state $(|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b)/\sqrt{2}$ is an entangled state, though it should be pointed out that it can be shown to be entangled by the application of other entanglement tests [1, 11]. This indicates that there is a need to find additional simple, and ideally, experimentally accessible conditions that can establish whether a state is entangled.

In this paper we provide a class of inequalities for detecting entanglement. These inequalities arise from examining uncertainty relations. The use of uncertainty relations to establish conditions for detecting entanglement has been pursued by Hofmann and Takeuchi [12] and by Guehne [13]. We begin by examining observables that are quadratic in the mode creation and annihilation operators. These observables were used previously to define sum and difference squeezing, forms of higher-order squeezing [14]. These quantities and their uncertainties are, in principle, measurable, so that the conditions we derive could be used in a laboratory to detect entanglement. We find that the conditions formulated in terms of these variables lead to a host of other conditions for detecting entanglement. Finally, we shall briefly discuss how some of these conditions can be extended to detect entanglement in systems consisting of more than two modes.

Consider two modes of the electromagnetic field, where a and $a^\dagger$ are the annihilation and creation operators of the first mode and b and $b^\dagger$ are the annihilation and creation operators of the second. We define the operators $L_1 = ab^\dagger + a^\dagger b$ and $L_2 = i(ab^\dagger - a^\dagger b)$. Operators, proportional to these, along with one proportional to the operator $L_3 = a^\dagger a + b^\dagger b$, form a representation of the su(2) Lie algebra, i.e., $J_i = L_i/2$ $(i = 1-3)$ satisfy the commutation relations $[J_k, J_m] = i\epsilon_{kmn}J_n$. Entanglement conditions expressed in terms of angular momentum operators have been derived by a number of authors [15]-[18]. It follows, on calculating the uncertainties of these variables and adding them, that

$$(\Delta L_1)^2 + (\Delta L_2)^2 = 2(\langle (N_a+1)N_b\rangle + \langle N_a(N_b+1)\rangle - 2|\langle ab^\dagger\rangle|^2), \tag{1}$$

where $N_a = a^\dagger a$ and $N_b = b^\dagger b$. Now suppose that the state we are considering is a product of a state in the a mode and another state in the b mode. Then the expectation values in the above expression factorize into products of a-mode and b-mode expectation values. We then have that

$$(\Delta L_1)^2 + (\Delta L_2)^2 = 2(\langle (N_a+1)\rangle\langle N_b\rangle + \langle N_a\rangle\langle (N_b+1)\rangle - 2|\langle a\rangle\langle b^\dagger\rangle|^2), \tag{2}$$

Noting that the Schwarz inequality implies that $|\langle a\rangle|^2 \leq \langle N_a\rangle$ and $|\langle b\rangle|^2 \leq \langle N_b\rangle$, we find that for a product state

$$(\Delta L_1)^2 + (\Delta L_2)^2 \geq 2(\langle N_a\rangle + \langle N_b\rangle). \tag{3}$$

This inequality can be extended to any separable state by using a result of Hofmann and Takeuchi [12]. For a density matrix $\rho = \sum_m p_m\rho_m$ and a variable S, we have that

$$(\Delta S)^2 \geq \sum_m p_m(\Delta S_m)^2, \tag{4}$$

where $(\Delta S_m)^2$ is the uncertainty of S calculated in the state ρ_m. If the original state ρ is separable, then all of the states ρ_m can be taken to be product states for

which the inequality in Eq. (3) holds. Then, Eq. (4) implies that Eq. (3) holds for the state ρ as well. Hence, Eq. (3) is valid for any separable state. It can be easily shown that Eq. (3) is violated for the Bell state $|\psi_{01}\rangle = (|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b)/\sqrt{2}$.

We can gain more insight if we consider the uncertainty relation obeyed by L_1 and L_2,

$$(\Delta L_1)(\Delta L_2) \geq |\langle N_a - N_b\rangle|. \tag{5}$$

This implies that

$$(\Delta L_1)^2 + (\Delta L_2)^2 \geq 2|\langle N_a - N_b\rangle|. \tag{6}$$

Comparing this result, which holds for any state, to Eq. (3), which holds for separable states, we see that the right-hand side of Eq. (6) is always less than or equal to that of Eq. (3). Consequently, there are states that violate Eq. (3) while satisfying Eq. (6), and the state in the previous paragraph is an example of such a state.

It is also worthwhile to see how the condition in Eq. (3)performs for a mixed state. Consider the state

$$\rho = s|\psi_{01}\rangle\langle\psi_{01}| + \frac{1-s}{4}P_{01}, \tag{7}$$

where $0 \leq s \leq 1$ and P_{01} is the projection operator onto thespace spanned by the vectors $\{|0\rangle_a|0\rangle_b, |0\rangle_a|1\rangle_b, |1\rangle_a|0\rangle_b, |1\rangle_a|1\rangle_b\}$. We find that $(\Delta L_1)^2 + (\Delta L_2)^2 = 3 - s - s^2$ and $2(\langle N_a\rangle + \langle N_b\rangle) = 2$, so that violation of the inequality in Eq. (3) shows that the state is entangled if $s^2 + s - 1 > 0$, or $1 \geq s > (\sqrt{5}-1)/2$.

An examination of the condition in Eq. (3) shows us that the state is entangled if

$$\langle N_a N_b\rangle < |\langle ab^\dagger\rangle|^2. \tag{8}$$

Note that the Schwarz inequality implies that

$$|\langle ab^\dagger\rangle|^2 \leq \langle N_a(N_b+1)\rangle, \tag{9}$$

so there are states that can satisfy the inequality in Eq. (8). This condition suggests that there is a family of similar conditions for detecting entanglement, where instead of considering the operator $ab^\dagger$ we consider instead $a^m(b^\dagger)^n$. For a pure product state we have that

$$|\langle a^m(b^\dagger)^n\rangle|^2 = |\langle a^m\rangle|^2|\langle b^n\rangle|^2 \leq \langle (a^\dagger)^m a^m\rangle\langle (b^\dagger)^n b^n\rangle, \tag{10}$$

or, because for a product state $\langle (a^\dagger)^m a^m\rangle\langle (b^\dagger)^n b^n\rangle = \langle (a^\dagger)^m a^m (b^\dagger)^n b^n\rangle$, it is also true that

$$|\langle a^m(b^\dagger)^n\rangle|^2 \leq \langle (a^\dagger)^m a^m (b^\dagger)^n b^n\rangle. \tag{11}$$

It is this relation that will lead to a generalization of the entanglement condition in Eq. (8), but before it does, we need to show that it holds for any separable state and not just for product states. Consider the density matrix for a general separable state given by $\rho = \sum_k p_k\rho_k$, where ρ_k is a density matrix corresponding to a pure product state, and p_k is the probability of ρ_k. The probabilities satisfy the condition $\sum_k p_k = 1$. Defining $A = a^m$ and $B = b^n$, we have that

$$\begin{aligned} |\langle AB^\dagger\rangle| &\leq \sum_k p_k|\mathrm{Tr}(\rho_k AB^\dagger)| \\ &\leq \sum_k p_k(\langle A^\dagger AB^\dagger B\rangle_k)^{1/2}, \end{aligned} \tag{12}$$

where $\langle A^\dagger AB^\dagger B\rangle_k = \mathrm{Tr}(\rho_k A^\dagger AB^\dagger B)$. We can now apply the Schwarz inequality to obtain

$$\begin{aligned} |\langle AB^\dagger\rangle| &\leq \left(\sum_k p_k\right)^{1/2}\left(\sum_k p_k\langle A^\dagger AB^\dagger B\rangle_k\right)^{1/2} \\ &\leq (\langle A^\dagger AB^\dagger B\rangle)^{1/2}, \end{aligned} \tag{13}$$

which shows that the inequality in Eq. (11) does indeed hold for all separable states. Therefore, we can conclude that a state is entangled if

$$|\langle a^m(b^\dagger)^n\rangle|^2 > \langle (a^\dagger)^m a^m (b^\dagger)^n b^n\rangle. \tag{14}$$

Let us now turn our attention to the variables $K_1 = ab + a^\dagger b^\dagger$ and $K_2 = i(a^\dagger b^\dagger - ab)$. One half times these operators along with one half times the operator the operator $K_3 = a^\dagger a - b^\dagger b$ form a representation of the su(1,1) Lie algebra. As before, we would like to find inequalities involving these variables that tell us whether a two-mode state is entangled or not. The strategy that we employed before, adding the uncertainties and assuming the expectation values can be factorized, leads to the inequality for product states $(\Delta K_1)^2 + (\Delta K_2)^2 \geq 2(\langle N_a\rangle + \langle N_b\rangle + 1)$. However, if we employ the uncertainty relation, $\Delta K_1\Delta K_2 \geq \langle N_a + N_b + 1\rangle$, we find that the above inequality holds for all states, and is therefore not useful for determining whether a state is entangled or not.

We can obtain something useful if we pursue a different path. The guiding idea is that the "eigenstates" (the reason for the quotation marks is that these states are, in general, not normalizable, and hence do not lie in the Hilbert space of two-mode states) of operators such as K_1 and K_2 are highly entangled. States whose uncertainty in one of these variables is small will be close to one of these eigenstates, and will also be entangled. Therefore, for a state to be separable, its uncertainty in one of these variables must be greater than some lower bound. What we shall show is that in the case of K_1 and K_2, that lower bound is 1.

In order to make the discussion more general, define the variable

$$K(\phi) = e^{i\phi}a^\dagger b^\dagger + e^{-i\phi}ab. \tag{15}$$

Note that $K(0) = K_1$ and $K(\pi/2) = K_2$. We then have that

$$\begin{aligned} (\Delta K(\phi))^2 &= \langle (a^\dagger b^\dagger - \langle a^\dagger b^\dagger\rangle)(ab - \langle ab\rangle)\rangle \\ &\quad + \langle (ab - \langle ab\rangle)(a^\dagger b^\dagger - \langle a^\dagger b^\dagger\rangle)\rangle \\ &\quad + e^{2i\phi}\langle (a^\dagger b^\dagger - \langle a^\dagger b^\dagger\rangle)^2\rangle \\ &\quad + e^{-2i\phi}\langle (ab - \langle ab\rangle)^2\rangle. \end{aligned} \tag{16}$$

We again employ the Schwarz inequality to give us

$$
\begin{aligned}
|\langle (ab-\langle ab\rangle)^2\rangle| \;\leq\; & [\langle (ab-\langle ab\rangle)(a^\dagger b^\dagger - \langle a^\dagger b^\dagger\rangle)\rangle \\
& \langle (a^\dagger b^\dagger - \langle a^\dagger b^\dagger\rangle)(ab-\langle ab\rangle)\rangle]^{1/2} \quad (17)
\end{aligned}
$$

This gives us that

$$
\begin{aligned}
(\Delta K(\phi))^2 \;\geq\; & [\langle (ab-\langle ab\rangle)(a^\dagger b^\dagger - \langle a^\dagger b^\dagger\rangle)\rangle^{1/2} \\
& -\langle (a^\dagger b^\dagger - \langle a^\dagger b^\dagger\rangle)(ab-\langle ab\rangle)\rangle^{1/2}]^2 \\
\geq\; & [(\langle (N_a+1)(N_b+1)\rangle - |\langle ab\rangle|^2)^{1/2} \\
& -(\langle N_a N_b\rangle - |\langle ab\rangle|^2)^{1/2}]^2. \quad (18)
\end{aligned}
$$

This inequality is valid for all states, but if the state is a product state this becomes

$$
\begin{aligned}
(\Delta K(\phi))^2 \;\geq\; & [(\langle (N_a+1)\rangle\langle (N_b+1)\rangle - |\langle a\rangle\langle b\rangle|^2)^{1/2} \\
& -(\langle N_a\rangle\langle N_b\rangle - |\langle a\rangle\langle b\rangle|^2)^{1/2}]^2. \quad (19)
\end{aligned}
$$

Now let us examine the quantity on the right-hand side of the above inequality. Setting $x=\langle N_a\rangle$, $y=\langle N_b\rangle$, and $z=|\langle ab\rangle|^2$, we want to find the minimum of the function

$$
F(x,y)=\sqrt{(x+1)(y+1)-z}-\sqrt{xy-z}, \quad (20)
$$

in the region $xy\geq z\geq 0$. By setting $\partial F/\partial x$ and $\partial F/\partial y$ equal to zero, we find that $F(x,y)$ has no local maxima or minima in the region of interest, so that the minimum of the function must lie on the boundary. This means we have to look at how F behaves on the curve $xy=z$ and as x and y go to infinity. On the curve $xy=z$ we find that

$$
F(x,z/x)=\left(x+\frac{z}{x}+1\right)^{1/2}\geq 1. \quad (21)
$$

Now let us consider what happens as $x,y\rightarrow\infty$. We first note that

$$
F(x,y)=\int_{xy-z}^{(x+1)(y+1)-z} du\frac{1}{2\sqrt{u}}\geq\frac{x+y+1}{2\sqrt{(x+1)(y+1)-z}}. \quad (22)
$$

Continuing, we find

$$
\begin{aligned}
F(x,y) \;\geq\; & \frac{x+y+1}{2\sqrt{(x+1)(y+1)}}=\frac{(x+1)+(y+1)-1}{2\sqrt{(x+1)(y+1)}} \\
\geq\; & \frac{1}{2}\left[\sqrt{\frac{x+1}{y+1}}+\sqrt{\frac{y+1}{x+1}}-\frac{1}{\sqrt{(x+1)(y+1)}}\right] \\
\geq\; & 1-\frac{1}{2\sqrt{(x+1)(y+1)}}. \quad (23)
\end{aligned}
$$

Therefore, we can conclude that as $x,y\rightarrow\infty$ we have that $F(x,y)\geq 1$. Finally, this gives us $(\Delta K(\phi))\geq 1$ for a product state, and the argument in [12] (see Eq. (4)) then implies that it is true for any separable state. Consequently, if for some state $(\Delta K(\phi))<1$, we can conclude it is entangled.

Both K_1 and K_2 are measurable. If the two modes are sent into a nonlinear crystal, to lowest order in the nonlinearity, the quadrature components of the mode corresponding to their sum frequency are proportional to K_1 and K_2 [14]. These quadratures can then be determined by means of homodyne measurements.

Let us exhibit a state for which $\Delta K_1<1$. Consider the state

$$
|\psi\rangle=\frac{1}{\sqrt{\eta}}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{\sqrt{2n+1}}|2n\rangle_a|2n\rangle_b, \quad (24)
$$

where $0\leq x<1$ and

$$
\eta=\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right). \quad (25)
$$

For this state we find that $\langle ab\rangle=0$, and

$$
\begin{aligned}
\langle\psi|a^2b^2|\psi\rangle \;=\; & -\frac{1}{\eta}\sum_{n=0}^{\infty}\frac{(2n+2)(2n+1)}{[(2n+3)(2n+1)]^{1/2}}x^{2n+1} \\
\langle\psi|(N_a+1)(N_b+1)|\psi\rangle \;=\; & \frac{1+x^2}{\eta(1-x^2)^2} \\
\langle\psi|N_aN_b|\psi\rangle \;=\; & \frac{3x^2-1}{\eta(1-x^2)^2}+1, \quad (26)
\end{aligned}
$$

which implies

$$
(\Delta K_1)^2=1+\frac{4x^2}{\eta(1-x^2)^2}-\frac{2}{\eta}\sum_{n=0}^{\infty}\frac{(2n+2)(2n+1)}{[(2n+3)(2n+1)]^{1/2}}x^{2n+1}. \quad (27)
$$

We can find a lower bound for the sum in the above equation, which gives us an upper bound for $(\Delta K_1)^2$. We obtain

$$
(\Delta K_1)^2\leq 1+\frac{4}{\eta}\left[-\frac{1}{\sqrt{3}}x+\frac{x^2[1-x(2-x^2)]}{(1-x^2)^2}\right]. \quad (28)
$$

Noting that near $x=0$ we have that η is approximately equal to $1+(x^2/3)$ we find that near $x=0$ the right-hand side of the above equation behaves like $1-(4x/\sqrt{3})$, so that ΔK_1 for this state can indeed be less than 1.

In analogy to what was done for the su(2) variables it is possible to find other relations that must be obeyed by separable states. For example, in the case of product states we have that

$$
|\langle ab\rangle|=|\langle a\rangle\langle b\rangle|\leq[\langle N_a\rangle\langle N_b\rangle]^{1/2}, \quad (29)
$$

and what we shall now do is show that this inequality is obeyed by all separable states. Therefore, a violation of this inequality implies that the state is entangled. In fact, we shall show that for any positive integers m and n, a separable state must satisfy the condition

$$
|\langle a^mb^n\rangle|\leq[\langle (a^\dagger)^ma^m\rangle\langle (b^\dagger)^nb^n\rangle]^{1/2}. \quad (30)
$$

Clearly Eq. (29) is a special case of Eq. (30).

As before, consider the density matrix of a general separable state $\rho = \sum_k p_k \rho_k$, where ρ_k is a density matrix corresponding to a pure product state, and p_k is the probability of ρ_k. Again, setting $A = a^m$ and $B = b^n$, we have that

$$\begin{aligned} |\langle AB\rangle|^2 &\leq \sum_{k,l} p_k p_l |\mathrm{Tr}(\rho_k AB)||\mathrm{Tr}(\rho_l B^\dagger A^\dagger)| \\ &\leq \sum_{k,l} p_k p_l (\langle A^\dagger A\rangle_k \langle B^\dagger B\rangle_k \langle A^\dagger A\rangle_l \langle B^\dagger B\rangle_l)^{1/2} \end{aligned} \quad (31)$$

In terms of the quantities $\langle A^\dagger A\rangle_k = \mathrm{Tr}(A^\dagger A \rho_k) = x_k$ and $\langle B^\dagger B\rangle_k = \mathrm{Tr}(B^\dagger B \rho_k) = y_k$, this inequality can be rewritten as

$$|\langle AB^\dagger\rangle|^2 \leq \sum_k p_k^2 x_k y_k + 2\sum_{k>l} p_k p_l (x_k y_k x_l y_l)^{1/2}. \quad (32)$$

Next we consider $\langle A^\dagger A\rangle\langle B^\dagger B\rangle = \sum_k p_k^2 x_k y_k + \sum_{k>l} p_k p_l (x_k y_l + x_l y_k)$. As $x_k y_l + x_l y_k \geq 2(x_k y_k x_l y_l)^{1/2}$, we see that the inequality in Eq. (30) holds for all separable states, i.e., if a state violates this inequality, it must be entangled.

Returning to the case $m = n = 1$, we have that for a general state $|\langle ab\rangle| \leq [\langle N_a + 1\rangle\langle N_b\rangle]^{1/2}$, which suggests that there are states that do violate the inequality in Eq. (29). An example of one that does is the two-mode squeezed vacuum state

$$|\psi\rangle = (1 - x^2)^{1/2} \sum_{n=0}^{\infty} x^n |n\rangle_a |n\rangle_b, \quad (33)$$

where $0 \leq x \leq 1$. For this state we find that $[\langle N_a\rangle\langle N_b\rangle]^{1/2} = x^2/(1-x^2) < x/(1-x^2) = |\langle ab\rangle|$, so that we conclude from Eq. (29) that this state is entangled.

We have derived a family of entanglement conditions for two-mode states. They enlarge the set of states that can be shown to be entangled by means of relatively simple conditions. Some of these conditions provide, in principle, measurable tests of entanglement, that is, all of the quantities appearing in the inequalities can be measured in the laboratory.

In closing, we point out that the methods employed here are not confined to demonstrating entanglement in two-mode states. To show this we briefly consider a three-mode example. A more thorough analysis will be left to future work. Consider three modes whose annihilation operators are a, b, and c. For a state that is a tensor product of individual states for each of the modes, we have that $|\langle ab^\dagger c^\dagger\rangle| = |\langle a\rangle\langle b\rangle\langle c\rangle| \leq (\langle N_a\rangle\langle N_b\rangle\langle N_c\rangle)^{1/2} = \langle N_a N_b N_c\rangle^{1/2}$. For a state that is completely separable in the three modes, that is one that can be expressed as $\rho = \sum_k p_k \rho_{ak} \otimes \rho_{bk} \otimes \rho_{ck}$, we find that $|\langle ab^\dagger c^\dagger\rangle| = \sum_k p_k |\langle a\rangle_k \langle b\rangle_k \langle c\rangle_k|$, which implies that

$$\begin{aligned} |\langle ab^\dagger c^\dagger\rangle| &\leq \sum_k p_k (\langle N_a N_b N_c\rangle_k)^{1/2} \\ &\leq (\sum_k p_k)^{1/2} (\sum_k p_k \langle N_a N_b N_c\rangle_k)^{1/2} \\ &\leq \langle N_a N_b N_c\rangle^{1/2}, \end{aligned} \quad (34)$$

where the next to last step follows from the Schwarz inequality. If a state is completely separable, it must obey this inequality, and, therefore, if the inequality is violated, the state will be entangled. An example of a state that does violate this inequality is given by $|\psi\rangle = (|1\rangle_a|0\rangle_b|0\rangle_c + |0\rangle_a|1\rangle_b|1\rangle_c)/\sqrt{2}$, which is a kind of GHZ state. In particular, for this state $\langle N_a N_b N_c\rangle = 0$, and $|\langle ab^\dagger c^\dagger\rangle| = 1/2$, which clearly violates the inequality $|\langle ab^\dagger c^\dagger\rangle| \leq \langle N_a N_b N_c\rangle^{1/2}$. Therefore, we see that the types of inequalities developed here can be extended to study the multipartite entanglement of continuous-variable systems.

Note added: After submission of this paper, publications on very similar topics by Agarwal and Biswas [19] and Shchukin and Vogel [20] have appeared.

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